Regular Variation and Smile Asymptotics

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Abstract

We consider risk-neutral returns and show how their tail asymptotics translate *directly* to asymptotics of the implied volatility smile, thereby sharpening Roger Lee's celebrated moment formula. The theory of regular variation provides the ideal mathematical framework to formulate and prove such results. The practical value of our formulae comes from the vast literature on tail asymptotics and our conditions are often seen to be true by simple inspection of known results.

1 Introduction

Consider risk-neutral returns X with cummulative distribution function F. We will impose a mild integrability condition on the right and left tail denoted by (IR), (IL) respectively. We write $\bar{F} = 1 - F$ and, if it exists, f for the probability density function of X. The class of regularly varying functions at $+\infty$ of index α is denoted by R_α. The reader not familiar with the theory of regular variation should think of positive multiples of x^α and harmless perturbations such as $\left(1 + x^{\alpha/2}\right)^2$, $x^\alpha \log x$ or $x^\alpha / (\log x)^3$. By convention, functions in R_α are (eventually) positive.

The purpose of this paper is to connect the tail behaviour of X to the wing behaviour of the Black-Scholes implied volatility, sharpening Roger Lee's celebrated moment formula [9]. From a mathematical point of view, the challenge is to relate the asymptotics of the distribution F to the asymptotics of a *non-linear transform*, namely the Black-Scholes implied volatility. From a financial point of view, we give further justification of volatility smile parametrizations seen in the industry and obtain new insights into the wing behaviour of a variety of models.

The normalized price of a Black-Scholes call with log-strike k is given by

$$c_{BS}(k,\sigma) = \Phi(d_1) - e^k \Phi(d_2)$$

with $d_{1,2}(k) = -k/\sigma \pm \sigma/2$. The implied volatility is the (unique) value $V(k)$ so that

$$c_{BS}(k, V(k)) = \int_k^\infty \left(e^x - e^k\right) dF(x) =: c(k).$$

We define the strictly decreasing function $\psi : [0,\infty] \to [0,2]$ by

$$\psi[x] = 2 - 4\left[\sqrt{x^2+x} - x\right].$$

Theorem 1 (Right-tail-wing formula) *Assume* $\alpha > 0$ *and*

$$\exists \epsilon > 0 : \mathbb{E}[e^{(1+\epsilon)X}] < \infty. \tag{IR}$$

Then

$$(i) \implies (ii) \implies (iii) \implies (iv),$$

where

$$-\log f(k) \in R_\alpha; \tag{i}$$

$$-\log \bar{F}(k) \in R_\alpha; \tag{ii}$$

$$-\log c(k) \in R_\alpha; \tag{iii}$$

and[1]

$$V(k)^2/k \sim \psi[-\log c(k)/k]. \tag{iv}$$

If (ii) holds then $-\log c(k) \sim -k - \log \bar{F}$ *and*

$$V(k)^2/k \sim \psi\left[-1 - \log \bar{F}(k)/k\right], \tag{iv'}$$

if (i) holds, then $-\log f \sim -\log \bar{F}$ *and*

$$V(k)^2/k \sim \psi[-1 - \log f(k)/k]. \tag{iv''}$$

Finally, if either $-\log f(k)/k$ *or* $-\log \bar{F}(k)/k$ *or* $-\log c(k)/k$ *goes to infinity as* $k \to \infty$ *then* $V^2(k)$ *behaves sublinearly. More precisely,*

$$V(k)^2/k \sim \frac{1}{-2\log f(k)/k} \, or \, \frac{1}{-2\log \bar{F}(k)/k} \, or \, \frac{1}{-2\log c(k)/k}. \tag{v}$$

We emphasize that (iv),(iv'),(iv") contain the *full asymptotics* of the implied volatility smile. For instance, we can see when $\limsup V(k)^2/k$ in Lee's moment formula is a genuine limit: $V(k)^2/k$ converges if and only if $-\log \bar{F}(k)/k$ converges to some limit θ. Note that our condition (IR) forces $\theta > 1$. Note also that in this case $-\log \bar{F} \in R_1$ so that condition (ii) is automatically satisfied.

[1] $g(k) \sim h(k)$ means $g(k)/h(k) \to 1$as $k \to \infty$.

In models without moment explosion (Black-Scholes, Merton's jump diffusion model, FMLS with $\beta = -1...$) the moment formula indicates sublinear behaviour of the implied variance, $\limsup V(k)^2/k = \lim V(k)^2/k = 0$, but yields no further information. In contrast, theorem 1 gives the *precise sublinear asymptotics.* For instance, in the Black-Scholes model $\log f_{BS}(k) \sim -k^2/\left(2\sigma^2\right)$ and (v) implies $V(k)^2 \sim \sigma^2$ in trivial agreement with the Black-Scholes flat volatility smile.

There is a similar result which, as $k \to \infty$, links $f(-k), F(-k)$, normalized out-of-the-money put prices $p(-k)$ and the implied volatility in the left wing.

Theorem 2 (Left-tail-wing formula) *Assume $\alpha > 0$ and*

$$\exists \epsilon > 0 : \mathbb{E}[e^{-\epsilon X}] < \infty. \tag{IL}$$

Then

$$\text{(i)} \implies \text{(ii)} \implies \text{(iii)} \implies \text{(iv)},$$

where

$$-\log f(-k) \in R_\alpha; \tag{i}$$

$$-\log F(-k) \in R_\alpha; \tag{ii}$$

$$-\log p(-k) \in R_\alpha; \tag{iii}$$

and

$$V(-k)^2/k \sim \psi\left[-1 - \log p(-k)/k\right]. \tag{iv}$$

If (ii) holds then $-\log p(-k) \sim k - \log F(-k)$ and

$$V(-k)^2/k \sim \psi\left[-\log F(-k)/k\right], \tag{iv'}$$

if (i) holds, then $-\log f(-k) \sim -\log F(-k)$ and

$$V(k)^2/k \sim \psi\left[-\log f(-k)/k\right]. \tag{iv''}$$

Finally, if either $-\log f(-k)/k$ or $-\log F(-k)/k$ or $-\log p(-k)/k$ goes to infinity as $k \to \infty$ then $V^2(-k)$ behaves sublinearly. More precisely,

$$V(-k)^2/k \sim \frac{1}{-2\log f(-k)/k} \text{ or } \frac{1}{-2\log F(-k)/k} \text{ or } \frac{1}{-2\log p(-k)/k}. \tag{v}$$

In conclusion, under mild integrability and regular variation conditions, tail asymptotics translate *directly* to asymptotics of the implied volatility smile. The practical value of formulae (iv'), (iv") comes from the vast literature on tail asymptotics and our conditions are often seen to be true by simple inspection of known results. The authors would like to thank Jim Gatheral, Roger Lee and Chris Rogers for related discussions. Financial support from the Cambridge Endowment for Research in Finance (CERF) is gratefully acknowledged.

2 Elements of Regular Variation Theory

Definition 3 *A positive real-valued measurable function f is regularly varying with index α, in symbols $g \in R_\alpha$ if*

$$\lim_{x\to\infty} \frac{g(\lambda x)}{g(x)} = \lambda^\alpha.$$

Functions in R_0 are called slowly varying.

The definition extends immediately to functions which are positive for x large enough. Recall that $g \sim h$ means $\lim_{x\to\infty} g(x)/h\,(x) = 1$. Trivially, $g \in R_\alpha$ and $h \sim g$ implies $h \in R_\alpha$. The following result can be found in the fine monograph [5, Thm 4.12.10, p255].

Theorem 4 (Bingham's Lemma) *Let $g \in R_\alpha$ with $\alpha > 0$ such that that e^{-g} is locally integrable at $+\infty$. Then*

$$-\log \int_x^\infty e^{-g(y)} dy \sim g\,(x)\,.$$

3 Right-Wing Smile Asympotics

Proof of Theorem 1. We first remark in presence of condition (IR), under either assumption (ii) or (iii) one must actually have $\alpha \geq 1$. Indeed,

$$\bar{F}\,(k) = \mathbb{P}[X > k] \leq e^{-(1+\epsilon)k}\mathbb{E}[e^{(1+\epsilon)X}] \implies -\log \bar{F}\,(k) \geq (1+\epsilon)\,k.$$

Assuming $-\log \bar{F} \in R_\alpha$ for $\alpha < 1$ would entail $-\log \bar{F}\,(k) \leq k^{\alpha'}$ for $\alpha' \in (\alpha, 1)$ and k large enough which contradicts the lower bound just established. Similarly,

$$c\,(k) = \mathbb{E}[(e^X - e^k)^+] \leq \mathbb{E}[e^X; X > k] \leq \mathbb{E}[e^X e^{\epsilon(X-k)}] = e^{-\epsilon k}\mathbb{E}[e^{(1+\epsilon)X}]$$

so that $-\log c\,(k) \geq \epsilon k$ and $-\log c \in R_\alpha$ can only happen with $\alpha \geq 1$.
(i) $\implies$ (ii): Use Binghan's lemma with the (eventually) positive function $g\,(k) = -\log f\,(k) \in R_\alpha$. (ii) $\implies$ (iii): Lemma 6. (iii) $\implies$ (iv): Lemma 7.The final statement on sublinear asymptotics follows from $\psi\,[x] \sim 1/\,(2x)$ as $x \to \infty$. Indeed,

$$\psi\,[x] = 2 - 4x\left(\sqrt{1 + 1/x} - 1\right) = 2 - 4x\left(\frac{1}{2x} - \frac{1}{8x^2} + O\left(x^{-3}\right)\right) \sim \frac{1}{2x}.$$

■

Lemma 5 *Assume (IR). Then the normalized call price with log-strike k is given by*

$$c\,(k) = \int_k^\infty e^x \bar{F}(x) dx. \tag{1}$$

Proof. Note that $\bar{F} = o\left(e^{-x}\right)$ since $\bar{F}(x) = \mathbb{P}[X > x] \leq e^{-(1+\epsilon)x}\mathbb{E}[e^{(1+\epsilon)X}]$. Integration by parts gives

$$\begin{aligned} c(k) &= -\int_k^\infty \left(e^x - e^k\right) d\bar{F}(x) \\ &= -\bar{F}(x)\left(e^x - e^k\right)\Big|_k^\infty + \int_k^\infty e^x \bar{F}(x)\, dx \end{aligned}$$

and the boundary contribution disappears. ■

Lemma 6 *Assume (IR) and that $-\log \bar{F} \in R_\alpha$ for some $\alpha \geq 1$. Then $-\log c \in R_\alpha$ and as $k \to \infty$,*

$$-\log c(k) \sim -k - \log \bar{F}(k). \tag{2}$$

Proof. Obviously $k \mapsto k \in R_1$ and $-\log \bar{F} \in R_\alpha, \alpha \geq 1$. We want to apply theorem 4 with $k \mapsto \varphi(k) \equiv -\log \bar{F}(k) - k$ to obtain (2) but need to be careful since in general the difference of two regularly varying functions may not be regularly varying. From the last proof we know that $-\log \bar{F}(k) \geq (1+\epsilon)\, k$. When $\alpha > 1$ then $\varphi(k)$ is immediately seen to be in R_α. When $\alpha = 1$ we can write $-\log \bar{F}(k) = k\, L(k)$ with slowly varying $L(k) \geq 1+\epsilon$. To see that $\varphi \in R_1$ it suffices to write

$$\frac{\varphi(\lambda k)}{\varphi(k)} = \frac{L(\lambda k) - 1}{L(k) - 1} = 1 + \frac{L(\lambda k)/L(k) - 1}{1 - 1/L(k)}$$

so that

$$\left|\frac{\varphi(\lambda k)}{\varphi(k)} - 1\right| \leq \left|\frac{L(\lambda k)/L(k) - 1}{1 - 1/L(k)}\right| \leq \frac{1+\epsilon}{\epsilon}\left|L(\lambda k)/L(k) - 1\right|$$

and this tends to zero as $k \to +\infty$ since $L \in R_0$. In either case, we can apply theorem 4 with $g = \varphi$ and obtain

$$-\log c(k) \overset{(1)}{=} -\log \int_k^\infty e^{x + \log \bar{F}(x)} dx \sim -\left(k + \log \bar{F}(k)\right) = \varphi(k).$$

Note that $\varphi(\cdot)$ was seen to be in R_α so that $-\log c(k) \sim \varphi(k)$ must also be in R_α. ■

Lemma 7 *Assume (IR) and $-\log c \in R_\alpha$ for some $\alpha \geq 1$. Then, as $k \to \infty$,*

$$\frac{\log c(k)}{k} = -\frac{k}{2V(k)^2} + \frac{1}{2} - \frac{V(k)^2}{8k} + O\left(\frac{\log k}{k}\right)$$

and

$$\frac{V(k)^2}{k} \sim \psi\left(\frac{-\log c(k)}{k}\right).$$

Proof. Appendix. ■

Remark 8 *In the preceding lemma the condition of regular variation can be replaced by the weaker*

$$\exists n \in \mathbb{N} : \lim \inf_{k \to \infty} V(k) k^n > 1.$$

4 Left-Wing Smile Asympotics

The proofs are similar and therefore omitted.

Lemma 9 *Assume (IL). Then the normalized put price with log-strike $-k$ is given by*

$$p(-k) = \int_{-\infty}^{-k} e^{x} F(x) dx = \int_{k}^{\infty} e^{-x} F(-x)\, dx.$$

Lemma 10 *Assume (IL) and that $-\log F(-k)$ is regularly varying. Then $-\log p(-k)$ is regularly varying as $k \to \infty$ and*

$$-\log p(-k) \sim k - \log F(-k). \tag{3}$$

Lemma 11 *Assume (IL) and that $-\log p(-k)$ is regularly varying. Then, as $k \to \infty$,*

$$\frac{V(-k)^2}{k} \sim \psi\left(-1 + \frac{-\log p(-k)}{k}\right).$$

5 Examples

In practice, one has $X = \log(S_T/F_T)$ where S_T denotes the risk-neutral stock price at time T and F_T is the time-T forward price. Also, $k = \log(K/F_T)$ and all quanitities f, F, c, p, V depend on time T and we set $V(k) = V(k,T) =: \sigma_{BS}(k,T)\sqrt{T}$, now calling $\sigma_{BS}(k,T)$ the implied volatility. It is worthwhile to spell of part (iv) of Theorem 1,

$$\sigma_{BS}^2(k,T)\, T/k \sim \psi\left[-\log c(k,T)/k\right].$$

Noting that $(S_T/F_T : T \geq 0)$ is a martingale and using convexity of the call payoff, it is easy to see, for k fixed, $c(k,T) = \mathbb{E}\left(e^X - e^k\right)^+$ is non-decreasing in T and so is $\psi\left[-\log c(k,T)/k\right]$. In fact, J. Gatheral and E. Reiner[2] point out independently that the *implied total variance* $\sigma_{BS}^2(k,T)\, T$ is non-decreasing in T, a consequence of monotonicity of the undiscounted Black-Scholes prices in $\sigma^2 T$. Thus, our asymptotic results respect the term structure of the volatility surface.
In the examples below, we will focus mainly on applications of the *right-tail-wing formula*, applications of the *left-tail-wing formula* being nearly identical.

5.1 Sanity Check: Black-Scholes Model

If σ denotes the Black Scholes volatility, the returns have a normal density with variance $\sigma^2 T$. Obviously then,

$$\log f_{BS}(k) \sim -k^2/\left(2\sigma^2 T\right)$$

and Theorem 1 implies $\sigma_{BS}(k,T)^2 \sim \sigma^2$ as $k \to \infty$ in trivial agreement $V \equiv \sigma$.

[2]Presentations at the Global Derivatives & Risk Management Conference 2004.

5.2 Barndorff-Nielsen's NIG Model

Here $X = X_T \sim NIG\,(\alpha, \beta, \mu T, \delta T)$. The moment generating function is given by

$$M\,(z) = \exp\left[T\left(\delta\left\{\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + z)^2}\right\} + \mu z\right)\right].$$

It is custom to write $\alpha = \sqrt{\beta^2 + \gamma^2}$ with $\gamma > 0$. From [4] and the references therein we have

$$f\,(k) \sim C\,|k|^{-3/2}\,e^{-\sqrt{\beta^2+\gamma^2}|k|+\beta k} \text{ as } k \to \pm\infty$$

and we see that $-\log f$ is regularly varying (with index 1). Moreover,

$$\log f\,(k)\,/k \to \left(-\sqrt{\beta^2 + \gamma^2} + \beta\right) \text{ as } k \to +\infty.$$

and from Theorem 1

$$\begin{aligned} \frac{\sigma_{BS}^2\,(k,T)\,T}{k} &\sim \psi\,(-1 - \log f\,(k)\,/k) \\ &\sim \psi\left(-1 + \sqrt{\beta^2 + \gamma^2} - \beta\right) \end{aligned}$$

in agreement with Lee's moment formula with critical moment $1+p^* = \sqrt{\beta^2 + \gamma^2} - \beta - 1 = \alpha - \beta$, as can be seen directly from the moment generating function, see [9].

5.3 Carr-Wu's Finite Moment Logstable Model

Here $X = X_T \sim L_\alpha\left(\mu T, \sigma T^{1/\alpha}, -1\right)$ where the law $L_\alpha\,(\theta, \sigma, \beta)$ has characteristic function

$$\mathbb{E}\left[e^{iuX}\right] = e^{iu\theta - |u|^\alpha \sigma^\alpha \left(1 - i\beta(\text{sig } u)\tan\frac{\pi\alpha}{2}\right)}.$$

with $\alpha \in (1,2], \theta \in \mathbb{R}, \sigma \geq 0, \beta \in [-1,1]$. From [6, page 10, equation (6)] and the references therein,

$$-\log \bar{F}\,(k) \sim k^{\frac{\alpha}{\alpha-1}} \times \left[T\alpha\sigma^\alpha\,|\sec\,(\pi\alpha/2)|\right]^{-1/(\alpha-1)} \text{ as } k \to \infty, \tag{4}$$

and from Theorem 1 we see that

$$\sigma_{BS}^2\,(k,T)\,T \sim k^{1-\frac{1}{\alpha-1}} \times \frac{1}{2}\left[T\alpha\sigma^\alpha\,|\sec\,(\pi\alpha/2)|\right]^{1/(\alpha-1)} \to 0 \text{ as } k \to \infty.$$

Note that in the limit $\alpha \uparrow 2$ the Black Scholes result is recovered. We also note that the moment generating function of $L_\alpha\,(\theta, \sigma, -1)$ exists for all positive z and is given by

$$M\,(z) = \exp\left[z\theta - (z\sigma)^\alpha \sec\left(\frac{\pi\alpha}{2}\right)\right];$$

Kasahara's exponential Tauberian theorem [5, p253] then gives immediately (4).

5.4 Merton's Jump Diffusion Model

As in the examples above, the return process $X.$ is Lévy with triplet (μ, σ^2, K) where K is λ (=intensity of jump) times a Gaussian measure with mean α and standard deviation δ describing the distribution of jumps. We now demonstrate how to proceed without explicit knowledge of the asymptotic tail. Set $X = X_T$ and note that $\mathbb{E}[\exp(zX)] < \infty$ for all z. Optimizing over z we get the tail estimate

$$\bar{F}(k) = \mathbb{P}[X > k] \le \inf_z e^{-zk}\mathbb{E}[\exp(zX)] = e^{K(z^*) - z^* k}$$

where $K(z) = \log \mathbb{E}[\exp(zX)]$ is the logarithmic mgf and $z^* = z^*(k)$ is determined from[3]

$$K'(z^*) = k.$$

For the Merton model,

$$K(z) = T\left\{z\mu + \frac{1}{2}z^2\sigma^2 + \lambda\left(e^{z\alpha + z^2\delta^2/2} - 1\right)\right\}$$

from which for $\delta > 0$ it is easy to see that

$$z^* = z^*(k) \sim \frac{\sqrt{2\log k}}{\delta}$$

so that $K(z^*) - z^* k \sim -z^* k$ and

$$\log \bar{F}(k) \lesssim -z^* k = -\frac{k}{\delta}\sqrt{2\log k}.$$

From the saddle point results of [7] or the Lévy tail estimates from [2] specialized to this example,

$$\log \bar{F}(k) \sim -\frac{k}{\delta}\sqrt{2\log k},$$

and Theorem 1 implies

$$\sigma_{BS}^2(k, T)\,T \sim \delta \times \frac{k}{2\sqrt{2\log k}}.$$

Note that this is independent of the mean jump size α provided $\delta > 0$. If $\delta = 0$ and $\alpha > 0$ a similar argument shows that $z^* = z^*(k) \sim \log k/\alpha$ and

$$\sigma_{BS}^2(k, T)\,T \sim \alpha \times \frac{k}{2\log k}. \tag{5}$$

Remark 12 (J. Gatheral) *A Poisson process with intensity λ has n jumps with probability $e^{-\lambda}\lambda^n/n! = e^{g(n)}$ with $g(n) \sim -n\log n$ by Stirling's formula. The time $T = 1$ Black-Scholes value of a digital is $\Phi(d_2)$ and $\log\Phi(d_2) \sim -k^2/2\sigma^2$. Identifying $k \sim n\alpha$ leads to*

$$-k^2/2\sigma^2 \sim -(k/\alpha)\log k \implies \sigma^2 \sim \alpha \times \frac{k}{2\log k}$$

in agreement with (5).

[3] Readers familiar with large deviation theory recognize the Fenchel-Legendre transform.

6 Further Examples and Discussion

Our tail wing formulae apply as soon as one has some (in fact: crude) asymptotic knowledge of the tail behaviour of the returns. As remarked in the introduction, for many models fine tail asymptotics are available in the literature and the smile asymptotics follow. In particular, the estimates of Albin-Bengtsson [1, 2] allow to use our results for the vast majority of exponential Lévy models [10]: NIG appears as a special case of generalized hyperbolic processes, Meixner processes as special cases of GZ processes. CGMY and Variance Gamma model are also covered. Tail estimates for stochastic volatility models appear in the literature, although one has to be careful with results obtained in short time regimes since the limits $T \to 0$ and $k \to \infty$ in general do not interchange[4]. We remark that condition (IR) rules out models for which *every* p^{th}-moment of the underlying, $p > 1$, explodes and a similar remark applies to condition (IL). Unfortunately, there are stochastic volatility models with this degenerate behaviour [3] and our (current) results do not apply.
When a moment generating function is known, one can often use an exponential Tauberian theorem to obtain log-tail estimates, as demonstrated in the Example 5.3. Also, the Fenchel-Legendre transform gives quick upper bounds which often turn out to be sharp, compare Example 5.4. We finally remark that some of the implications in Theorem 1 and Theorem 2 can be reversed under Tauberian conditions but we shall not pursue this here.

7 Appendix

Proof of Lemma 7. The following bounds for the normal distribution function Φ are well-known and can be obtained by integration by parts (or other methods),

$$\frac{e^{-x^2/2}}{\sqrt{2\pi}x}(1-\frac{1}{x^2}) \le \Phi(-x) \le \frac{e^{-x^2/2}}{\sqrt{2\pi}x}, \quad x > 0.$$

Using assumption (IR),

$$E[(e^X - e^k)_+] \le E[e^X; X > k] \le E[e^X e^{\epsilon(X-k)}] = e^{-\epsilon k} E[e^{(1+\epsilon)X}],$$

we see that $c\,(k)$ is exponentially small in k, and this implies that

$$a := \limsup_{k\to\infty} \frac{V(k)^2}{k} < 2 \qquad (6)$$

In particular, both

$$d_1(k) = -\frac{k}{V(k)} + \frac{V(k)}{2}, \;\; d_2(k) = -\frac{k}{V(k)} - \frac{V(k)}{2}$$

[4]The wrong wing behaviour in Hagan's SABR formula, based on short time asymptotics, is a warning example.

will then be negative for k large enough so that we will be able to use the bounds on Φ. From the Black-Scholes formula for a normalized call with log-strike k,

$$c\,(k) = \Phi(d_1) - e^k \Phi(d_2),$$

we obtain

$$e^{-d_1^2/2}\left(-\frac{1}{d_1}\right)\left(1-\frac{1}{d_1^2}\right) - e^k e^{-d_2^2/2}\left(-\frac{1}{d_2}\right) \le \sqrt{2\pi}c\,(k)$$

and

$$\sqrt{2\pi}c\,(k) \le e^{-d_1^2/2}\left(-\frac{1}{d_1}\right) - e^k e^{-d_2^2/2}\left(-\frac{1}{d_2}\right)\left(1-\frac{1}{d_2^2}\right).$$

Because $d_1^2/2 = d_2^2/2 - k$, this simplifies to

$$e^{-d_1^2/2}\left[-\frac{1}{d_1}\left(1-\frac{1}{d_1^2}\right)+\frac{1}{d_2}\right] \le \sqrt{2\pi}c\,(k) \le e^{-d_1^2/2}\left[-\frac{1}{d_1}+\frac{1}{d_2}\left(1-\frac{1}{d_2^2}\right)\right].$$

We now define $\epsilon_1 = \epsilon_1\,(k)$ by

$$\log c\,(k) = -\frac{d_1^2}{2} + \epsilon_1(k)$$

and will show $|\epsilon_1\,(k)| = O\,(\log k)$. To start, we note the bounds

$$-\frac{1}{d_1}\left(1-\frac{1}{d_1^2}\right)+\frac{1}{d_2} \le \sqrt{2\pi}e^{\epsilon_1(k)} \le -\frac{1}{d_1}+\frac{1}{d_2}\left(1-\frac{1}{d_2^2}\right). \tag{7}$$

From (6) we know that $V(k) \le \sqrt{2}k$, hence

$$-d_2 = \frac{k}{V(k)} + \frac{V(k)}{2} \ge \sqrt{k/2} \to \infty \text{ as } k \to \infty.$$

Therefore, for k large enough,

$$\sqrt{2\pi}e^{\epsilon_1(k)} \le -\frac{1}{d_1}.$$

In fact, (6) gives $a' \in (0,2)$ s.t. $V\,(k)^2 < a'k$ for all k large enough. From the definition of d_1 we then see that

$$a'''\sqrt{k} := \sqrt{k}\left(\frac{1}{\sqrt{a'}} - \frac{\sqrt{a'}}{2}\right) < -d_1\,(k)$$

with $a''' > 0$ because $a' \in (0,2)$. Hence

$$\sqrt{2\pi}e^{\epsilon_1(k)} \le \frac{1}{a'''\sqrt{k}}$$

and we see that $\epsilon_1(k) \to -\infty$ as $k \to \infty$ and we only need a bound on $-\epsilon_1(k)$. We start by showing

$$\exists n \in \mathbb{N} : \liminf_{k\to\infty} V(k)k^n > 1. \tag{8}$$

To see this note that for k large enough $\epsilon_1 < 0$ so that

$$-\log c(k) = \frac{d_1^2}{2} - \epsilon_1(k) \geq \frac{d_1^2}{2}$$

By assumption $-\log c \in R_\alpha$ for $\alpha \geq 1$ so there exists $L \in R_0$ so that $-\log c(k) = k^\alpha L(k)$ and

$$k^\alpha L(k) \geq \frac{d_1^2}{2} = \frac{1}{2}\left(-\frac{k}{V(k)} + \frac{V(k)}{2}\right)^2 \geq \frac{k^2}{2V(k)^2} - \frac{k}{2}$$

so that

$$\frac{k^2}{V(k)^2} \leq 2k^\alpha L(k) + k \equiv k^\alpha \tilde{L}(k)$$

where $\tilde{L} \in R_0$ and $V(k)^2 k^{\alpha-2} = 1/\tilde{L}(k)$ and (8) holds with any integer $n > (\alpha-2)^+$.

We return to establish a bound on $-\epsilon_1(k)$. We already have lower and upper bounds on the implied volatility, namely (replace n by $n+1$ if needed)

$$k^{-2n} < V(k)^2 < a'k \text{ with } a' < 2 \text{ and } k \text{ large enough,}$$

which we can use to bound $d_{1,2}$, namely

$$\begin{aligned} -d_1 &< k^{1+n} - \frac{1}{2k^n} \leq k^{1+n} \\ -d_2 &< k^{1+n} + \frac{\sqrt{2k}}{2k} \leq 2k^{1+n} \end{aligned} \tag{9}$$

(at least for $k \geq 1$). In order to derive an upper bound for $-\epsilon_1$ recall that a lower bound on $\sqrt{2\pi}e^{\epsilon_1}$ was given in (7) by

$$\begin{aligned} \epsilon_2(k) &\equiv -\frac{1}{d_1}\left(1 - \frac{1}{d_1^2}\right) + \frac{1}{d_2} \\ &= \frac{d_1 - d_2}{d_1 d_2} + \frac{1}{d_1^3} = \frac{V}{d_1 d_2} + \frac{1}{d_1^3} \\ &= \frac{V^2 d_1^2 + V d_2}{V d_1^3 d_2} \\ &= \frac{\left(-k + V^2/2\right)^2 + \left(-k - V^2/2\right)}{V d_1^3 d_2} =: (*) \end{aligned}$$

We already noted the existence of $\alpha' \in (0,2)$ such that $x := V^2/2 < \beta k < k$ with $\beta = \alpha'/2 < 1$ for k large enough. Noting that, for k fixed, the function

$$[0, \beta k] \ni x \mapsto (-k + x)^2 + (-k - x)$$

is strictly decreasing on $[0, \beta k]$ we can get a lower bound on ϵ_2 as follows,

$$\begin{aligned}
(*) \quad &\geq \frac{(-k+\beta k)^2 + (-k-\beta k)}{d_1^3 d_2 V} \\
&= \frac{k^2 (1-\beta)^2 - k(1+\beta)}{d_1^3 d_2 V} \\
&\geq \frac{\frac{1}{2}k^2 (1-\beta)^2}{d_1^3 d_2 \sqrt{\beta k}} \quad \text{(for } k \text{ large enough)} \\
&\geq \frac{\frac{1}{2}k^2 (1-\beta)^2}{(-d_1)^3 (-d_2) \sqrt{\beta k}} \\
&= \frac{\frac{1}{2}k^2 (1-\beta)^2}{(k^{1+n})^3 (2k^{1+n}) \sqrt{\beta k}} \quad \text{(use (9))} \\
&= \gamma k^{2-4(1+n)-1/2} \text{ with } \gamma > 0.
\end{aligned}$$

Therefore,

$$\sqrt{2\pi} e^{\epsilon_1(k)} > \epsilon_2(k) > \gamma k^{-5/2-4n} > 0$$

and after taking logarithms

$$-\epsilon_1(k) < (4n + 5/2)\log k + C$$

for some constant $C = C(\gamma)$. In particular,

$$|\epsilon_1(k)| = O(\log k).$$

Recall that ϵ_1 was defined by

$$\log c(k) = -\frac{d_1^2}{2} + \epsilon_1(k) = \frac{k}{2} - \frac{k^2}{2V^2} - \frac{V^2}{8} + \epsilon_1(k)$$

and after dividing by k we have,

$$\left| \frac{\log c(k)}{k} - \left(\frac{1}{2} - \frac{k}{2V^2} - \frac{V^2}{8k} \right) \right| = O(\log k / k)$$

which tends to zero as $k \to \infty$ so that

$$\frac{\log c(k)}{k} \sim \frac{1}{2} - \frac{k}{2V^2} - \frac{V^2}{8k}.$$

We can rearrange the above equations as a quadratic equation in V^2/k,

$$-\frac{\log c(k)}{k} + \frac{\epsilon_1(k)}{k} = \frac{1}{2V^2/k} + \frac{V^2/k}{8} - \frac{1}{2}.$$

One solution is given by

$$\frac{V^2}{k} = \psi \left(-\frac{\log c(k)}{k} + \frac{\epsilon_1(k)}{k} \right) \sim \psi \left(-\frac{\log c(k)}{k} \right),$$

while the other is rejected as it violates (6). ■

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